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THE STRONGLY ADEQUATE MATRICES OF THE FORM OF PRODUCT OF LINDENBAUM'S MATRICES*

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In this paper two problems are considered. The first one concerns existence of matrices strongly adequate for certain logics and is connected with some results presented in [6]. All matrices constructed here have the same form, more precisely, they are products of Lindenbaum's matrices. The second problem concerns Maksimova's principle of separating variables (cf. [5]).

Let $\underline{S} = \langle S, \rightarrow, +, \cdot, \sim \rangle$ be the algebra of formulas free-generated by the set $At = \{p, q, r, p_1, p_2, \dots\}$, and let $\underline{S}_p = \langle S_p, \rightarrow, +, \cdot, \sim \rangle$ be the subalgebra generated by the set p . The algebras $\underline{S}^I = \langle S^I, \rightarrow \rangle$ and $\underline{S}^P = \langle S^P, -, +, \cdot \rangle$ are the implicational and positive fragments of the language \underline{S} , respectively. $At(X)$ denotes the set of all propositional variables occurring in all formulas in X .

The symbols r_0, r_{ad} stand for the modus ponens rule and the adjunction rule, respectively. The structural consequence generated by a matrix $M = \langle A, D \rangle$ is denoted by \vec{M} . The symbol $\prod_{t \in T} M_t$ stands for the product of matrices $\{M_t\}_{t \in T}$. We recall that $Y \in Sat(M)$ iff there exists a valuation $v : At \rightarrow A$ such that $h^v(Y) \subseteq D$. Then one can prove (cf. [4], [9]) that:

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$$(*) \quad \overrightarrow{\prod_{t \in T} M_t(Y)} = \begin{cases} \bigcap_{t \in T} \vec{M}_t(Y) & \text{if } Y \in \text{Sat}(M_t) \text{ for every } t \in T \\ S & \text{if } Y \notin \text{Sat}(M_t) \text{ for some } t \in T \end{cases}$$

We say that matrix M is strongly adequate for a structural consequence Cn iff $Cn = \vec{M}$.

Let (A) denote the following condition:

$$(A) \rightarrow [(\psi \rightarrow \Gamma) \rightarrow (\varphi \rightarrow \Gamma), (\psi \rightarrow \Gamma) \rightarrow (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \Gamma)], \varphi \rightarrow \varphi \in Cn(0), \text{ and } r_0 \text{ is derivable in } Cn \text{ (i.e. } \psi \in Cn(\{\varphi, \varphi \rightarrow \psi\})), \text{ all } \varphi, \psi, \Gamma \in S^I \text{ } [S^P],$$

and (B) denote the following formula:

$$(B) ((p \rightarrow p) \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p) \in Cn(0)$$

THEOREM 1. *For any structural consequence Cn on the language \underline{S}^I , if (A), (B) hold for Cn , then $Cn = \overrightarrow{\prod_{X \subseteq S^I} M_X}$, where M_X is Lindenbaum's matrix of the form $M_X = \langle S^I, Cn(X), \rightarrow \rangle$.*

Now, we shall formulate an analogon of T1 in the language \underline{S}^P . To do that, let Cn be a consequence on \underline{S}^P while (C) denotes the following condition:

$$(C) \varphi \rightarrow \varphi + \psi, \psi \rightarrow \varphi + \psi, (\varphi \rightarrow \Gamma) \cdot (\psi \rightarrow \Gamma) \rightarrow (\varphi + \psi \rightarrow \Gamma), \\ \varphi \cdot \psi \rightarrow \varphi, \varphi \cdot \psi \rightarrow \psi, (\varphi \rightarrow \psi) \cdot (\varphi \rightarrow \Gamma) \rightarrow (\varphi \rightarrow \psi \cdot \Gamma) \in Cn(0) \\ \text{and the adjunction rule } r_{ad} \text{ is derivable in } Cn, \text{ all } \varphi, \psi, \Gamma \in S^P.$$

THEOREM 2. *For any structural consequence Cn on the language \underline{S}^P , if (A), (B), (C), hold for Cn , then $Cn = \overrightarrow{\prod_{X \subseteq S^P} M_X}$, $M_X = \langle S^P, Cn(X), -, +, \cdot \rangle$.*

It may be desirable to notice that the scope of T2 is sufficiently large. Namely, the premises of this theorem hold for the relevant logics E, R, RM , the intuitionistic logic and all their extensions (in the positive language).

What about \underline{S} ? It is easily seen that the analogon of T2 need not be valid for a structural consequence on language \underline{S} . This can be shown by an example of a consequence Cn_E based on Belnap's and Anderson's system of entailment in its structural version (i.e. without the rule of substitution), (cf. 1).

THEOREM 3. $Cn_E < \overrightarrow{\prod_{X \subseteq S} M_X}$, where M_X is Lindenbaum's matrix of the form $M_X = \langle S, Cn_E(X), -, +, \cdot, \sim \rangle$.

However some consequences on \underline{S} have strongly adequate matrices of the form of a product of the some Lindenbaum's matrices. Theorem 4 and 6 show it.

THEOREM 4. Let Cn be a structural consequence on \underline{S} . Then, the following conditions are equivalent:

- (i) if $\varphi \in Cn(X \cup S_p)$ and $p \notin At(X \cup \{\varphi\})$ then $\varphi \in Cn(X)$; for all $\varphi \in S, X \subseteq S, p \in At$,
- (ii) if $\varphi \in Cn(X \cup Y)$ and $At(Y) \cap At(X \cup \{\varphi\}) = 0$ then $\varphi \in Cn(X)$; for all $\varphi \in S, X, Y \subseteq S$,
- (iii) $Cn = \overrightarrow{\prod_{X \subseteq S; p \in At} M_{X,p}}$, where $M_{X,p}$ is Lindenbaum's matrix of the form $M_{X,p} = \langle S, Cn(X \cup S_p), \rightarrow, +, \cdot, \sim \rangle$ for $p \notin At(X)$.

We write $\prod M_{X,p}$ instead of $\prod_{X \subseteq S; p \in At} M_{X,p}$.

Directly from T4 we get:

COROLLARY 1. For every structural and consistent consequence Cn on \underline{S} the following conditions are equivalent:

- (i) if $\varphi \in Cn(X \cup S_p)$ and $p \notin At(X \cup \{\varphi\})$ then $\varphi \in Cn(X)$, for all $\varphi \in S, X \subseteq S, p \in At$;
- (ii) $Cn(S_p) \neq S$ for all $p \in At$ and there exists a strongly adequate matrix for Cn .

Now, let R be the set of theorems of the well-known system of relevant implication (cf. [2]) and let Cn_R be the structural consequence based on the set R and the rules: r_0, r_{ad} .

THEOREM 5. (Maksimova [5]) Let Cn be one of the consequences Cn_E, Cn_R . If $\varphi \in Cn(X \cup \{\psi\})$ and $At(\psi) \cap At(X \cup \{\varphi\}) = 0$, then $\varphi \in Cn(X)$; for all $\varphi, \psi \in S, X \subseteq S$.

It is easy to observe that the condition (i) of T4 and Maksimova's Theorem are equivalent for the consequences Cn_E, Cn_R . With this argument, from T4 and T5 we can immediately deduce (cf. [6]).

THEOREM 6. If Cn is one of the consequences Cn_E , Cn_R , then $Cn = \overrightarrow{\prod} M_{X,P}$; where is such as in T4.

Finally, let us examine a question similar to the previous one, that is to Maksimova's Theorem. For this purpose we shall consider only such finite structural consequences Cn on $\underline{S}^I[\underline{S}^P]$ that $Cn \leq Cn_2$, (Cn_2 is a structural version of the classical logic on $\underline{S}^I[\underline{S}^P]$). Then, by Łoś-Suszko's theorem in [3] (see also [8]) on existence of a strongly adequate matrix and by T1 [T2], we can easily deduce:

COROLLARY 2. If $(A), (B), [(A), (B), (C)]$ in \underline{S}^P hold for Cn (Cn finite, $Cn \leq Cn_2$), then for all $X, Y \subseteq \underline{S}^I[\underline{S}^P]$ and $\varphi \in \underline{S}^I[\underline{S}^P]$, $\varphi \in Cn(X)$, whenever $\varphi \in Cn(X \cup Y)$ and $At(Y) \cap At(X \cup \{\varphi\}) = 0$.

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